Table 1 Inverse and feedforward matrix solutions

$$S_{11}=\begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -0.119 & 0.297 & -1.37\times10^{-8} & 0.812 & 0.0 \\ 0.00263 & 0.00444 & 0.388 & 0.0217 & -8.31\times10^{-8} \\ -0.00221 & -0.00931 & 0.0 & 1.02 & 0.0 \\ 13.4 & -23.5 & -3.75\times10^{-7} & 3.82 & 0.0128 \end{bmatrix}$$

$$S_{12}=0$$

$$S_{21}=\begin{bmatrix} 0.00126 & -0.00371 & -0.103 & -0.0112 & 0.0 \\ 0.481 & -0.283 & -39.5 & -3.19 & 1.72\times10^{-4} \end{bmatrix}$$

$$S_{22}=\begin{bmatrix} 0.0186 & 1.062\times10^{-5} \\ 3.52\times10^{-9} & 0.319 \end{bmatrix}$$

$$\Omega_{11}=\begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -0.7924712 & 0.0 & 42.476485 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & -0.0147849 & 0.0 & 0.7924712 & 0.0 \\ 340.13605 & -21.552343 & 0.0 & 1155.2063 & 0.0 \end{bmatrix}$$

$$\Omega_{21}=\begin{bmatrix} 0.0 & 0.00883 & -0.128 & -0.685 & 0.0 \\ 8.50 & -0.539 & 0.0 & 28.9 & 0.025 \end{bmatrix}$$

It can be shown, using results from Ref. 6, that Kawahata eliminates the last term in Eq. (9) by choosing some elements in K_x so that

$$(\Omega_{21}-K_x\Omega_{11})\nu=0 \qquad (10)$$

The minimal realization of the generalized state space compensator in Eq. (5) (the dynamical equation is also sometimes referred to as a descriptor system) cancels transmission zeroes of the plant not common between the plant and model. If some of the elements in K_x can be chosen to satisfy Eq. (10), then some of the closed-loop eigenvalues of the plant equal all of the *finite* plant transmission zeroes as discussed by Kawahata. Plant transmission zero locations do not always satisfy desirable handling qualities specifications.

Reworking the example in Ref. 1 using Eqs. (4) and (9) where the plant is a Beechcraft Model 65 and the model to be followed is that of a Boeing 747 yields the results in Table 1. Using Table 1, and K_x from Table 2 in Ref. 1, the values for $S_{21}-K_xS_{11}$ and $S_{22}-K_xS_{12}$ are in agreement with K_{xm} and K_{um} in Table 2 of Ref. 1 to three significant digits. Note that S_{12} is the null matrix, causing the minimal realization of ν to be $\nu=0$. The last term in Eq. (9) is zero for any value of K_x, a feature of this example that is also pointed out by Kawahata.

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[1] Kawahata, N., "Model-Following System with Assignable Error Dynamics and Its Application to Aircraft," *Journal of Guidance and Control,* Vol. 3, Nov.-Dec. 1980, pp. 508-516.

[2] Broussard, J.R. and O'Brien, M.J., "Feedforward Control to Track the Output of a Forced Model," *IEEE Transactions on Automatic Control,* Vol. AC-25, Aug. 1980, pp. 851-854.

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Reply by Author to J.R. Broussard and L.E. Mabius

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I WISH to thank Mr. Broussard and Mr. Mabius for their Comment on my work[1] and for drawing attention to their Ref. 2. When my manuscript was prepared, Ref. 2 was not yet available. Even so, the two papers complement each other.

In Ref. 1, a practically oriented design technique was stressed rather than a more theoretically oriented approach to the model following system. The appearance of time

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derivatives of the model input u_m which are often not available has been intentionally excluded by the conditions $\sigma_k \leq \sigma_{mk}$ [Eq. (5) in Ref. 1] and $|B^*| \neq 0$. As will be mentioned later, the intention to eliminate $\dot{u}_m$ is further supported by the proposed feedback gains that yield some closed loop poles in common with the plant transmission zeros if any.

The model following control[1] provides a realization of the precompensator $G(s)$ in Eq. (4).[1] Hence, as is pointed out in the Comment with the aid of Ref. 3, neither the common zeros nor the common poles of $Y/U(s)$ and $Y_m/U_m(s)$ affect $G(s)$ since they cancel each other on both sides of Eq. (4). When unstable plant transmission zeros are not in common with zeros of the model, no approach can exactly handle such a model following system since $G(s)$ must effectively contain an inverse system of the plant. Although it is not explicitly stated in the Comment, the recommended approach[2] also requires the stability of plant transmission zeros. Otherwise, the precompensator ν-dynamics with an arbitrary input $\dot{u}_m$ become unstable because the eigenvalues of Ω_{11} are the inverses of the plant transmission zeros. This fact is demonstrated by a simple example:

Plant:

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} u \qquad (1)$$

$$y = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix} x$$

Model:

$$\dot{x}_m = \begin{bmatrix} -2.5 & 0 \\ 0 & -4 \end{bmatrix} x_m + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u_m \qquad (2a)$$

$$y_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_m \qquad (2b)$$

The above plant has an unstable transmission zero at $\lambda = 1$. The feedforward model following control law is computed along the line of the Comment and Ref. 2. In particular, the matrix Ω_{11} that determines the stability of the precompensator to the derivative input $\dot{u}_m$ [Eq. (5) in the Comment)] becomes

$$\Omega_{11} = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 4 & -4 \\ 0 & 3 & -3 \end{bmatrix}$$

whose eigenvalues are 1, 0, and 0 which are the inverses of the plant transmission zeros. It can be shown that the nonsingular transformation of ν, $Tw = \nu$, by T

$$T = \begin{bmatrix} 2 & 0 & 1 \\ -4 & 1 & 1 \\ -3 & 1 & 1 \end{bmatrix}$$

reduces the ν-dynamics of Eq. (5) in the Comment to be

$$\dot{w}_1 = w_1 - (2/3.5, -3/5)\dot{u}_m \qquad w_2 = w_3 \equiv 0$$

The first element of the transformed w is unstable and so is ν. Thus, the feedforward model following control proposed by the Comment and Ref. 2 is not practically realizable either, even if $\dot{u}_m$ were available.

Mr. Broussard and Mr. Mabius raised an additional requirement for the model following control that no transmission zero of the plant equals an eigenvalue of the model,[2] which was not mentioned by Ref. 1. However, a counterexample will be given to show that such a requirement is not necessary even for a feedforward realization of model following control. The approach recommended by the Comment, which claims to give a more general solution to the model following problem, may be somewhat more restrictive in this regard although this case may be rare in practice. Consider a plant given by the same state equation as Eq. (1) and the following output equation

$$y = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} x$$

This plant has a stable transmission zero at $\lambda = -2.5$. A desired model is assumed to be given by Eqs. (2a) and (2b). Then,

$$\frac{Y}{U}(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{-1}{s+3} \\ \dfrac{1}{s+1} & \dfrac{1}{s+2} \end{bmatrix} \qquad \frac{Y_m}{U_m}(s) = \begin{bmatrix} \dfrac{1}{s+2.5} & 0 \\ 0 & \dfrac{1}{s+4} \end{bmatrix} = \frac{X_m}{U_m}(s)$$

Obviously, the plant transmission zero equals an eigenvalue of the model. The approach suggested by the Comment cannot handle this example since there is some difficulty for the existence and the uniqueness of a series expansion of feedforward gains S_{ij}. However, the model following can be accomplished by a feedforward control as long as the compensator has the following function:

$$U_f(s) = [Y/U(s)]^{-1} [Y_m/U_m(s)] U_m(s)$$

$$U_f(s) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \frac{U_m(s)}{2} - \begin{bmatrix} \dfrac{s+3.25}{s+2.5} & \dfrac{3.5s+8}{s+4} \\ \dfrac{-0.25}{s+2.5} & \dfrac{1.5s+4}{s+4} \end{bmatrix} \frac{U_m(s)}{2(s+2.5)}$$

A feedforward realization of the above may be given by

$$U_f(s) = \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix} U_m(s) - \begin{bmatrix} 0.5 & 1.75 \\ 0 & 0.75 \end{bmatrix} X_m(s)$$

$$- \begin{bmatrix} 0.75 & -0.75 \\ -0.25 & 0.25 \end{bmatrix} \frac{X_m(s)}{2(s+2.5)}$$

which consists of the model input, the model state, and the filtered model state but not of the time derivatives of u_m. Needless to say, this example can be handled by the approach of Ref. 1 without any difficulty.

The author completely agrees with the Comment of Mr. Broussard and Mr. Mabius that a particular state feedback is not essential for the model following control. It is a matter of the choice of the realization of the compensator $G(s)$ as is stated in Ref. 1. A good combination of feedforward and feedback controls is preferable in practice. The Comment introduced a feedforward realization for some class of the problem, and suggested an interesting relation between the feedforward and the composite (feedforward/back) realizations. Consider a plant with an arbitrary negative feedback gain matrix L:

$$\dot{x}_c = (A - BL)x_c + Bu_f \qquad u = -Lx_c + u_f \qquad y_c = Cx_c$$

Assuming that no transmission zero coincides with model eigenvalues, define matrices Ω and S_{ij} for the open-loop system (A,B,C) as in Ref. 2. Then, the feedforward control with an inverse of the closed-loop system, $U_f(s) = [Y_c/U_f(s)]^{-1}[Y_m/U_m(s)]\ U_m(s)$, is shown to be generated by the same equation as in the Comment:

$$U_f(s) = (S_{21}+LS_{11})X_m(s) + (S_{22}+LS_{12})U_m(s)$$

$$+ (\Omega_{21}+L\Omega_{11})\nu(s)$$

$$\nu(s) \triangleq (I-s\Omega_{11})^{-1}S_{12}\dot{U}_m(s)$$

This shows how an arbitrary feedback gain L reflects in the feedforward gains. A little manipulation using relations among Ω, S_{ij}, and (A,B,C) yields the following quite interesting expression for the above $U_f(s)$:

$$U_f(s) = [B^{*-1}(A_m^* + KM_m) + (L-K_x)S_{11}]X_m(s)$$

$$+ [B^{*-1}B_m^* + (L-K_x)S_{12}]U_m(s) + (L-K_x)\Omega_{11}\nu(s)$$

where

$$K_x \triangleq B^{*-1}(A^* + KM)$$

Matrices A^*, B^*, M, K, K_x and $(\cdot)_m$ are all defined in Ref. 1.

The above expression suggests that if $L=K_x$, there is no need to have the time derivatives of u_m, and the control law becomes the same as that offered by Ref. 1. When the plant has no transmission zero, there always exists a unique matrix K to satisfy $L=K_x$ for an arbitrary feedback gain matrix L since M is nonsingular for an observable system. The matrix K is not necessarily of block-diagonal form as is noted in Ref. 1. On the other hand, when there is any (stable) transmission zero the matrix M is, in general, singular. The feedback gain matrix L to eliminate the time derivatives of u_m or ν is *not completely arbitrary* since the matrix K to satisfy $L=K_x$ may not exist. Thus, it has been made clear that the feedback gains offered in Ref. 1 are quite significant to avoid $\dot{u}_m$ in the model following control.

Since, in practice, the time derivatives of u_m are often not available, the system designer may wish to have a feedback control with the gain $L=K_x$. However, some of the transmission zeros may be close to the origin and the imaginary axis, so that the transient output error response may persist and the closed-loop response to system noise (e.g., aircraft gust response) be considerably dominant. If that is the case, one may opt for a feedforward realization with a tight feedback loop ($L \neq K_x$) at the expense of generating the time derivatives of u_m. Thus, in practical situations with stable transmission zeros, a compromise between the feedforward[2] and/or the feedback[1] realizations of the model following control may be called for.

References

[1] Kawahata, N., "Model-Following System with Assignable Error Dynamics and Its Application to Aircraft," *Journal of Guidance and Control*, Vol. 3, Nov.-Dec. 1980. pp. 508-516.

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Comment on "Time-Optimal Orbit Transfer Trajectory for Solar Sail Spacecraft"

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Introduction

IN REF. 1, Jayaraman discusses minimum-time heliocentric transfers between the Earth's orbit and the orbit of Mars using a solar sail propulsion system. The orbits of these planets are assumed to be circular and coplanar. Planetary masses are neglected. Jayaraman applies the calculus of variations to search for minimum-time trajectories for two values of solar sail characteristic thrust acceleration.[2] Jayaraman's solutions differ from those obtained by Kelley[3,4] and Zhukov and Lebedev.[5] His transfer times are about 10% larger and his sail orientation histories significantly different. Jayaraman asserts that he has found the minimum-time solutions and that the earlier, shorter transfer times in Refs. 3-5 were probably for trajectories which do not accurately satisfy the required boundary conditions. We show here that the solutions in Refs. 3-5 (and also in Ref. 2) are correct and that a transversality condition of variational calculus has been applied incorrectly in Ref. 1.

Reference 1 also discusses the merits of solar sailing for carrying out interplanetary space missions, relative to electric propulsion, in particular. We argue here that the orbit transfer problems considered contain too many simplifications to allow a realistic comparison of these advanced propulsion technologies and note that a number of more sophisticated studies of solar sailing missions were carried out in the late 1970's.

Conditions of Optimality

The normalized equations of motion for a planar heliocentric transfer by means of a solar sail spacecraft are

$$\mathrm{d}x_1/\mathrm{d}t = x_2 \tag{1}$$

$$\mathrm{d}x_2/\mathrm{d}t = x_3^2/x_1 - 1/x_1^2 + \beta\cos^3 u/x_1^2 \tag{2}$$

$$\mathrm{d}x_3/\mathrm{d}t = -x_2x_3/x_1 + \beta\cos^2 u \sin u/x_1^2 \tag{3}$$

where t, x_1, x_2, x_3, and u denote time, radial distance of the spacecraft from the sun, radial velocity, circumferential velocity, and angle between the outwardly directed normal to

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